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LETTER TO THE EDITOR

A long-range Domany–Kinzel model of directed percolation

T C Li and Z Q Zhang
Institute of Physics, Chinese Academy of Sciences, Beijing, China

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Abstract. A long-range Domany–Kinzel model proposed by Wu and Stanley is solved using random-walk formulations. In this model, for every site (i, j) in a two-dimensional lattice there is a directed bond present from site (i, j) to $(i+1, j)$ with probability one. There are also $m+1$ directed bonds present from (i, j) to $(i-k, j+1)$, $k=-1, 0, 1, 2, 3\ldots m-1$ with respective probabilities p_{k+1} where m is any positive integer. An exact expression is obtained to determine the critical percolation angle θ_c for any distribution of p_n. The system percolates in the region $\theta_c > \theta \geqslant 0$ with probability one and zero outside this region where θ is the angle measured from the x axis. If we let m go to infinity and p_n varies as $p_1 n^{-s}$, we find that when $s \leqslant 2$, $\theta_c = \pi$. A closed form expression of θ_c is obtained for $s \geqslant 2$. When m is large but finite, θ_c is also obtained for the following two distributions: (a) $p_n = a/(a+n)$ with $a>0$, (b) $p_n = \beta/m$ and $\beta > 0$.

Directed percolation has been the focus of much attention in the past few years. This is not only because it forms a new universality class with anisotropic scaling but also because of its close relationship to the Reggeon field theory in high-energy physics and the Markov process with breaking, recombination and absorption that occur in chemistry and biology, etc. (For a review see Kinzel (1982).) In two dimensions, various methods, like series expansion (Blease 1977, Essam and De'Bell 1981, De'Bell and Essam 1983), Monte Carlo (Kertesz and Vicsek 1980, Dhar and Barma 1981), and finite-size scaling (Kinzel and Yeomans 1981, Domany and Kinzel 1981) have been performed and much progress has been achieved.

Recently, Domany and Kinzel (1981, hereafter referred to as DK) proposed a particular two-dimensional solvable model. For every site (i, j) in a square lattice, they considered the case when a horizontal directed bond is present from (i, j) to $(i+1, j)$ with probability $p_H = 1$ and a vertical directed bond is present from (i, j) to $(i, j+1)$ with probability $p_V = p$ (figure 1). Let $P(\boldsymbol{R}, p)$ be the probability that a site $\boldsymbol{R} = (N, L)$ is connected to the origin. DK found that for large R there exists a critical percolation angle $\theta_c(p)$ such that the system percolates in the region $\theta_c > \theta \geqslant 0$ with probability one and zero outside this region, where $\theta = \tan^{-1}(L/N)$. When $\theta > \theta_c$ they also found that the correlation length exponent $\nu = 2$. More recently, Wu and Stanley (1982, hereafter referred to as WS) were able to reformulate the Domany–Kinzel model in terms of a random-walk problem using the fact that a unique path can be singled out for each percolation configuration. This reformulation greatly simplifies the problem. WS not only solved the case of a triangular lattice but also suggested that a long-range Domany–Kinzel model can be worked out using these random-walk formulations.

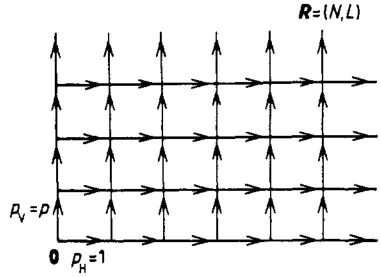

Figure 1. Directed percolation in a square lattice with $p_H = 1$ and $p_V = p$.

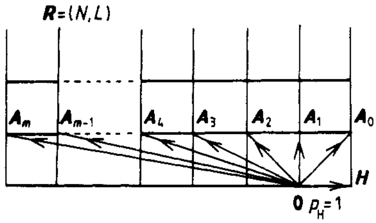

Figure 2. Directed percolation with $p_H = 1$. There are other $m+1$ directed bonds $\boldsymbol{OA}_0$, $\boldsymbol{OA}_1, \ldots, \boldsymbol{OA}_m$ being present with respective probabilities $p_0, p_1, \ldots, p_m$.

For this long-range model, in addition to a directed bond being present from site (i, j) to $(i+1, j)$ with probability one for every site (i, j), there are $m+1$ directed bonds present from (i, j) to $(i-k, j+1)$, $k=-1, 0, 1, 2, \ldots, m-1$ with respective probabilities p_{k+1} where m is any positive integer (figure 2). From figure 2, it is easily seen that the largest θ_c one can have is $\tan^{-1}[1/(1-m)]$. This occurs only when $p_m = 1$. When m becomes infinite and $\lim_{m\to\infty} p_m = 1$ we certainly have $\theta_c = \pi$. However, this is a rather uninteresting case. What is more interesting is the following. Under what circumstances can one still have $\theta_c = \pi$ even when $\lim_{m\to\infty} p_m = 0$? If we let p_n decay like $p_1 n^{-s}$, will a critical s_c exist such that when $s \leq s_c$, $\theta_c = \pi$? In this case, the fact that θ_c approaches π must be attributed to the cumulative contributions of an infinite number of bonds. This problem has something similar to the one-dimensional Ising model with a long-range interaction $J(n)$ which decays like n^{-s} (Dyson 1969), or the one-dimensional percolation model with long-range bonds p_n which decay like n^{-s} (Zhang *et al* 1983). For both cases, the ordered states stemming from long-range interactions are found at $s_c = 2$.

In this letter, we work out this long-range Domany–Kinzel model explicitly. Some exact results are presented. In particular, we find that s_c is again equal to 2. When m is large but finite, we also discuss two specific distributions of p_n.

First we consider the case when m is finite (figure 2). Following WS, we single out a unique path for every percolating configuration from $\boldsymbol{O}$ to $\boldsymbol{R}$ by the following procedure. Starting from $\boldsymbol{O}$ one traverses along the $\boldsymbol{OA}_m$ bond if it is present. If the $\boldsymbol{OA}_m$ bond is not present then one traverses along the $\boldsymbol{OA}_{m-1}$ bond if it is present, etc. If all the $m+1$ bonds ($\boldsymbol{OA}_m$, $\boldsymbol{OA}_{m-1} \ldots$, $\boldsymbol{OA}_1$ and $\boldsymbol{OA}_0$) are absent then one traverses horizontally along the $\boldsymbol{OH}$ bond. This process is repeated at each new site until one reaches $\boldsymbol{R}$. Thus the unique path singled out by this procedure is the left-most path connecting $\boldsymbol{O}$ and $\boldsymbol{R}$ in every percolating configuration. A random-walk problem can then be formulated. At every site, a walker can only walk along the following $m+2$ directed bonds $\boldsymbol{OA}_m$, $\boldsymbol{OA}_{m-1} \ldots$, $\boldsymbol{OA}_1$, $\boldsymbol{OA}_0$ and $\boldsymbol{OH}$ with respective probabilities $\tilde{p}_m, \tilde{p}_{m-1}, \ldots, \tilde{p}_1, \tilde{p}_0$ and $\tilde{p}_H$, where $\tilde{p}_m = p_m$, $\tilde{p}_{m-1} = q_m p_{m-1}$, $\tilde{p}_{m-2} = q_m q_{m-1} p_{m-2}, \ldots, \tilde{p}_0 = q_m q_{m-1} \ldots q_1 p_0$, $\tilde{p}_H = q_m q_{m-1} \ldots q_1 q_0$ and $q_i = 1 - p_i$ for all $i = 0, 1, 2, \ldots, m$. Using the standard method (Montroll 1964), the probability $W_{n,L-1}$ that a walker will reach site $(n, L-1)$ from $(0, 0)$ is given by

$$W_{n,L-1} = \frac{1}{(2\pi)^2} \int\int_{-\pi}^{\pi} \frac{\mathrm{d}\varphi_1\, \mathrm{d}\varphi_2 \exp[-\mathrm{i}n\varphi_1 - \mathrm{i}(L-1)\varphi_2]}{1 - (\tilde{p}_H \exp(\mathrm{i}\varphi_1) + \tilde{p}_0 \exp[\mathrm{i}(\varphi_1+\varphi_2)] + \tilde{p}_1 \exp(\mathrm{i}\varphi_2) + \tilde{p}_2 \exp[\mathrm{i}(\varphi_2 - \varphi_1)] + \ldots + \tilde{p}_m \exp\{\mathrm{i}[\varphi_2 - (m-1)\varphi_1]\})}. \tag{1}$$

Following WS, the probability $P(\boldsymbol{R},\boldsymbol{p})$ ($\boldsymbol{p}=\{p_0,p_1,\ldots,p_m\}$) that $\boldsymbol{R}$ is connected to the origin is related to $W_{n,L-1}$ by

$$P(\boldsymbol{R},\boldsymbol{p})=(1-q_mq_{m-1}\ldots q_1q_0)\sum_{n=(1-m)(L-1)}^{N-1}W_{n,L-1}+(1-q_mq_{m-1}\ldots q_2q_1)W_{N,L-1}+\ldots$$
$$+(1-q_mq_{m-1})W_{N+m-2,L-1}+(1-q_m)W_{N+m-1,L-1}. \tag{2}$$

The above equation is just the straightforward extension of equation (4) of WS. To evaluate (1) and (2), we follow exactly the same analysis as given by WS. Here, we will skip all the derivations and only present the following results. Let $\alpha=N/L$ ($\boldsymbol{R}=(N,L)$); we find that when R is large

$$\begin{aligned} P(\boldsymbol{R},\boldsymbol{p})&=1, &&\alpha>\alpha_c(\boldsymbol{p}),\\ &=0, &&\alpha<\alpha_c(\boldsymbol{p}),\\ &=\tfrac{1}{2}, &&\alpha=\alpha_c(\boldsymbol{p}),\end{aligned}\tag{3}$$

where

$$\alpha_c(\boldsymbol{p})=\mathcal{N}(\boldsymbol{p})/\mathcal{D}(\boldsymbol{p})\tag{4}$$

with

$$\mathcal{N}(\boldsymbol{p})=q_m+q_mq_{m-1}+\ldots+q_mq_{m-1}\ldots q_2q_1-(m-1),\tag{5}$$

$$\mathcal{D}(\boldsymbol{p})=1-q_mq_{m-1}q_{m-2}\ldots q_2q_1q_0.\tag{6}$$

The critical percolation angle $\theta_c(p)$ is thus equal to $\tan^{-1}[\alpha_c^{-1}(\boldsymbol{p})]$. When $p_m=1$ ($q_m=0$), from (4)–(6), we indeed obtain $\theta_c=\tan^{-1}[1/(1-m)]$. For $\alpha<\alpha_c(\boldsymbol{p})$, we also find that $P(R,p)\sim\exp(-R/\xi(p))$ with $\xi(\boldsymbol{p})\sim[\alpha_c(\boldsymbol{p})-\alpha]^{-2}$. So, the critical exponent $\mu=2$ remains unchanged if any finite number of further neighbour directed bonds are added to the system. This is consistent with the universality concept.

The results (4)–(6) can be seen directly using the random-walk displacements. In the random-walk formulations, every walk from $\boldsymbol{O}$ to $\boldsymbol{R}=(N,L)$ represents the leftmost path of certain configurations in which all the sites (M,L) with $M>N$ are connected to the origin while those with $M<N$ are disconnected from the origin. If $\langle x\rangle$ and $\langle y\rangle$ are respectively the average random-walk displacements along the x and y directions, when R is large, $\alpha_c(\boldsymbol{p})$ is thus $\langle x\rangle/\langle y\rangle$. More explicitly,

$$\langle x\rangle=\sum_{i=0}^{m}(1-i)\tilde{p}_i+\tilde{p}_{\rm H}=1-F(m),\tag{7}$$

$$\langle y\rangle=\sum_{i=0}^{m}\tilde{p}_i=1-\tilde{p}_{\rm H},\tag{8}$$

where

$$F(m)=\sum_{i=1}^{m}i\tilde{p}_i.\tag{9}$$

Using the relations $\tilde{p}_i=p_iq_{i+1}\ldots q_m$, it is easy to see from (4)–(6) and (7)–(9) that $\langle x\rangle$ and $\langle y\rangle$ are respectively the same functions as $\mathcal{N}(p)$ and $\mathcal{D}(p)$ of (5) and (6).

In the following, we first discuss three special distributions of p_i from which the properties of $F(m)$ can be determined exactly.

(i) $p_i=p_1/i^s$, $i=1,2,\ldots$ ($m\to\infty$)

For this distribution, the interesting question is to find a critical s_c such that $\theta_c=\pi$

when $s \leq s_c$. Since $\mathscr{D}(p)$ of (6) (or $\langle y \rangle$ of (8)) is bounded by $0 < \mathscr{D}(\boldsymbol{p}) \leq 1$, $\theta_c = \pi$ or $\theta_c < \pi$ according to whether $F(m)$ in equation (7) diverges or converges as $m \to \infty$. This can be studied using the following theorem (the proof will be given in the appendix).

Theorem. If $\Sigma_{i=1}^{m} p_i$ converges then $F(m)$ converges or diverges according to whether $\Sigma_{i=1}^{m} ip_i$ converges or diverges as $m \to \infty$.

Since $\Sigma_{i=1}^{m} p_i$ converges for $s > 1$, it follows from the theorem that $\theta_c < \pi$ for $s > 2$ and $\theta_c = \pi$ for $1 < s \leq 2$. So, we find exactly $s_c = 2$.

(ii) $p_i = a/(a+i)$, $a > 0$, $i = 1, 2, \ldots, m$

In this case, one can prove easily by induction that $F(m)$ has the expression

$$F(m) = a \sum_{i=1}^{m} \frac{i(i+1)\ldots m}{(a+i)\ldots(a+m)} = \frac{am}{a+1}. \tag{10}$$

When m is large, the function $\mathscr{D}(\boldsymbol{p})$ of (6) can be written as

$$\mathscr{D}(\boldsymbol{p}) = 1 - q_0 \prod_{i=1}^{m} \frac{i}{i+a} \approx 1 - q_0 a \Gamma(a) m^{-a} \tag{11}$$

where the Euler formula of the gamma function has been used (Abramowitz and Stegun 1970). From (7), (10) and (11), we find, to the leading order in m, $\theta_c = \tan^{-1}[-(1+a)/am]$ and $\theta_c \to \pi$ as $m \to \infty$.

(iii) $p_i = \beta/m$, $\beta > 0$, $i = 1, 2, \ldots, m$

This is a mean-field-like model (Wu 1982). In this case $F(m)$ can be evaluated directly and has the expression

$$F(m) = \sum_{i=1}^{m} \frac{i\beta}{m}\left(1 - \frac{\beta}{m}\right)^{m-i} = m - \left(\frac{m}{\beta} - 1\right)\left[1 - \left(1 - \frac{\beta}{m}\right)^{m}\right]. \tag{12}$$

$\mathscr{D}(\boldsymbol{p})$ of (6) is simply

$$\mathscr{D}(\boldsymbol{p}) = 1 - q_0(1 - \beta/m)^m. \tag{13}$$

In the large m limit, from (7), (12) and (13), we find to the leading order in m, $\theta_c = \tan^{-1}\{\beta[1 - q_0 \exp(-\beta)]/[m(1 - \beta - \exp(-\beta))]\}$ and $\theta_c \to \pi$ as $m \to \infty$.

Finally, when both m and $F(m)$ are large, we will derive a general expression of $F(m)$ for an arbitrary distribution of p_i. From (5) and (7), $F(m)$ can be written as

$$F(m) = (1 - q_m) + (1 - q_m q_{m-1}) + \ldots + (1 - q_m q_{m-1} \ldots q_2 q_1). \tag{14}$$

Using the identity

$$\begin{aligned} 1 - q_m q_{m-1} \ldots q_{m-i} &= p_{m-i} + p_{m-i+1} q_{m-i} \\ &\quad + p_{m-i+2} q_{m-i+1} q_{m-i} + \ldots + p_m q_{m-1} q_{m-2} \ldots q_{m-i+1} q_{m-i} \end{aligned} \tag{15}$$

equation (14) can be put in a recursive form

$$F(m) = \sum_{n=1}^{m} n p_n - \sum_{n=1}^{m-1} p_{n+1} F(n). \tag{16}$$

If p_i decays slowly enough that $F(m)$ becomes very large when m is large, (16) can be replaced by an integral equation, with x substituting m,

$$F(x)=\int_1^x yp(y)\mathrm{d}y-\int_1^{x-1} p(y+1)F(y)\mathrm{d}y. \tag{17}$$

Differentiating both sides of (17) with respect to x, we have

$$F'(x)=xp(x)-p(x)F(x-1). \tag{18}$$

In both cases when $F(x)$ is convergent or $F(x)$ is divergent in powers of x (as the cases discussed above), $F(x-1)$ can be replaced by $F(x)$ when x is large. So, we have

$$F'(x)=xp(x)-p(x)F(x). \tag{19}$$

(19) can be integrated by the standard method, leading to the expression

$$F(x)=\exp\left(-\int_1^x p(y)\mathrm{d}y\right)\int_1^x yp(y)\exp\left(\int_1^y p(z)\mathrm{d}z\right)\mathrm{d}y. \tag{20}$$

From (20), one can determine $F(x)$ for any distribution of p_i. As an example, we consider again the case (i) when p_i decays as p_1/i^s. Substituting $p(z)=p_1z^{-s}$ into (20), after integrations, we find

$$\lim_{x\to\infty} F(x)\equiv F_1(p_1,s)=\frac{p_1}{s-1}\Gamma\left(\frac{s-2}{s-1}\right)\gamma^*\left(\frac{s-2}{s-1}\frac{p_1}{s-1}\right) \tag{21}$$

where $\Gamma(a)$ and $\gamma^*(a,z)$ are respectively the gamma and incomplete gamma function (Abramowitz and Stegun 1970). Since $\gamma^*(a,z)$ is an analytic function of a and z, while $\Gamma(a)$ has a simple pole at $a=0$, from $s-2=0$, we find $s_c=2$. Equation (21) is valid when s is greater than but near 2 such that $F(m\to\infty)$ is large. $\mathscr{D}(p)$ of (6) in the $m\to\infty$ limit now becomes

$$\lim_{m\to\infty}\mathscr{D}(\boldsymbol{p})=1-(1-p_0)Z(p_1,s) \tag{22}$$

where

$$Z(p_1,s)=\prod_{n=1}^{\infty}\left(1-\frac{p_1}{n^s}\right). \tag{23}$$

Taking the logarithm of $Z(p_1,s)$, we have

$$-\ln Z(p_1,s)=-\sum_{n=1}^{\infty}\ln\left(1-\frac{p_1}{n^s}\right)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{p_1^m}{mn^{ms}}. \tag{24}$$

Replacing the double sum in (24) by double integrals, after integrations, we find

$$Z(p_1,s)=\exp\{E_1(-\ln p_1)-\exp[(\ln p_1)/s]E_1[[(1-S)\ln p_1]/s]\} \tag{25}$$

where $E_1(a)$ is the exponential integral (Abramowitz and Stegun 1970). So when $s\geqslant 2$, α_c has the expression

$$\alpha_c(p_0,p_1,s)=[1-F_1(p_1,s)]/[1-(1-p_0)Z(p_1,s)] \tag{26}$$

where $F_1(p_1,s)$ and $Z(p_1,s)$ are given by (21) and (25) respectively. From α_c one can obtain θ_c.

If we use the special distributions of cases (ii) and (iii) in (20), after integrations, we again obtain the same results as (10) and (12) to the leading order in m.

The authors are much indebted to Professor F Y Wu for introducing and suggesting this problem. They are also very grateful to the referee, Professor Essam, for pointing out the theorem and many useful comments and suggestions.

Appendix. Proof of the theorem

The convergence condition of the theorem is easily seen by using the relations $\tilde{p}_i < p_i$. In order to study the divergence condition, the following inequality is used:

$$F(m) = \sum_{i=1}^{m-1} ip_i \prod_{j=i+1}^{m} q_j + mp_m > \left(\sum_{i=1}^{m-1} ip_i\right)\left(\prod_{j=2}^{m} q_j\right) = \left(\sum_{i=1}^{m-1} ip_i\right)\left(\prod_{j=2}^{m} (1-p_j)\right).$$

If $\Sigma_{i=1}^{m} p_i$ converges, it can be shown that $\Pi_{j=2}^{m} (1-p_j)$ also converges as $m \to \infty$ (Arfken 1970). From the above inequality, it follows that $F(m)$ diverges as $m \to \infty$ whenever $\Sigma_{i=1}^{m} ip_i$ diverges.

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